Heat Equation

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Here we will study the equations

$$\Delta u = f$$

and

$$\begin{cases} \partial_t u - \Delta u & = f \\ u(x,0) & = u_0(x) \end{cases}$$

on either $\mathbb{R}^n$ or on manifolds. On $\mathbb{R}^n$ we will take $\Delta = \sum_i \partial_i^2$.

1 Harmonic, Subharmonic and Superharmonic functions on $\mathbb{R}^n$

Let's begin by proving the following important formulas. Throughout: Ω will be an open subset of $\mathbb{R}^n$ and $\mathcal{H}^{n-1}(dy)$ will denote the surface measure of the relevant $n-1$-dimensional surface in $\mathbb{R}^n$. Also, whenever I write $\overline{X}$ for X a subset of $\mathbb{R}^n$, I mean that $\overline{X}$ is the closure of X in $\mathbb{R}^n$. So if I write $\overline{X} \subseteq \Omega$ I mean that the closure of X in $\mathbb{R}^n$ is contained in Ω, not that the closure of X in Ω is contained in Ω (which is always true).

Lemma 1.1. *Let $u : \Omega \to \mathbb{R}$ be a C^2 function and suppose $\overline{B}_r(x) \subseteq \Omega$. Then*

$$\frac{d}{dr}\left(r^{1-n}\int_{\partial B_r(x)} u(y)\mathcal{H}^{n-1}(dy)\right) = r^{1-n}\int_{B_r(x)} (\Delta u)(y) d^n y.$$

Proof. We compute:

$$\begin{aligned} \frac{d}{dr}\left(r^{1-n}\int_{\partial B_r(x)} u(y)\mathcal{H}^{n-1}(dy)\right) &= \frac{d}{dr}\int_{\partial B_1(0)} u(x+rz)\mathcal{H}^{n-1}(dz) \text{ via } y = x+rz \\ &= \int_{\partial B_1(0)} (\nabla u)(x+rz)\cdot z\mathcal{H}^{n-1}(dz) \\ &= \int_{B_1(0)} r\,\mathrm{div}(\nabla u)(x+rz) d^n z \text{ by divergence theorem} \\ &= r^{1-n}\int_{B_r(x)} (\Delta u)(y) d^n y \text{ via } y = x+rz. \end{aligned}$$

□

As a corollary, we obtain the following important result concerning harmonic, subharmonic and superharmonic functions.

Corollary 1.2. *Let $u : \Omega \to \mathbb{R}$ be a C^2 function, $x \in \Omega$ and denote*

$$\varphi(r) := \frac{1}{\mathcal{H}^{n-1}(S^{n-1})} r^{1-n}\int_{\partial B_r(x)} u(y)\mathcal{H}^{n-1}(dy).$$

*If $\Delta u = 0$ (i.e. u is harmonic) then $\varphi(r)$ is constant. If $\Delta u \geq 0$ (u is **subharmonic**) then $\varphi(r)$ is monotonically increasing. If $\Delta u \leq 0$ (u is **superharmonic**) then $\varphi(r)$ is monotonically decreasing. In particular:*

$$\begin{aligned}
&\text{If } \Delta u = 0 \text{ then } u(x) = \frac{1}{\mathcal{H}^{n-1}(S^{n-1})} r^{1-n} \int_{\partial B_r(x)} u(y)\mathcal{H}^{n-1}(dy)\\
&\text{If } \Delta u \geq 0 \text{ then } u(x) \leq \frac{1}{\mathcal{H}^{n-1}(S^{n-1})} r^{1-n} \int_{\partial B_r(x)} u(y)\mathcal{H}^{n-1}(dy)\\
&\text{If } \Delta u \leq 0 \text{ then } u(x) \geq \frac{1}{\mathcal{H}^{n-1}(S^{n-1})} r^{1-n} \int_{\partial B_r(x)} u(y)\mathcal{H}^{n-1}(dy).
\end{aligned}$$

Proof. By 1.1 it follows that $\varphi'(r) = 0$ (respectively $\varphi'(r) \geq 0$ or $\varphi'(r) \leq 0$) if $\Delta u = 0$ (respectively $\Delta u \geq 0$ or $\Delta u \leq 0$). Therefore, it suffices for us to prove that

$$\lim_{r \searrow 0} \varphi(r) = u(x).$$

Towards this end, let $\epsilon > 0$. Since u is continuous at x, there exists a $\delta > 0$ such that

$$\text{if } y \in \Omega \text{ satisfies } |x - y| < \delta \text{ then } |u(x) - u(y)| < \epsilon.$$

Then, for $r < \delta$ we have:

$$y \in \partial B_r(x) \implies |x - y| < \delta$$

and so for $r < \delta$ we have

$$\begin{aligned}
|u(x) - \varphi(r)| &= \left| u(x) - \frac{1}{\mathcal{H}^{n-1}(\partial B_r(x))} \int_{\partial B_r(x)} u(y)\mathcal{H}^{n-1}(dy) \right|\\
&= \left| \frac{u(x)}{\mathcal{H}^{n-1}(\partial B_r(x))} \int_{\partial B_r(x)} 1\mathcal{H}^{n-1}(dy) - \frac{1}{\mathcal{H}^{n-1}(\partial B_r(x))} \int_{\partial B_r(x)} u(y)\mathcal{H}^{n-1}(dy) \right|\\
&= \frac{1}{\mathcal{H}^{n-1}(\partial B_r(x))} \left| \int_{\partial B_r(x)} u(x)\mathcal{H}^{n-1}(dy) - \int_{\partial B_r(x)} u(y)\mathcal{H}^{n-1}(dy) \right|\\
&\leq \frac{1}{\mathcal{H}^{n-1}(\partial B_r(x))} \int_{\partial B_r(x)} |u(x) - u(y)|\mathcal{H}^{n-1}(dy)\\
&\leq \frac{1}{\mathcal{H}^{n-1}(\partial B_r(x))} \int_{\partial B_r(x)} \epsilon\mathcal{H}^{n-1}(dy)\\
&\leq \epsilon.
\end{aligned}$$

Therefore, by the $\epsilon - \delta$ definition of the limit, we indeed have

$$\lim_{r \searrow 0} \varphi(r) = u(x).$$

□

Sometimes it will be useful to have versions of the above result where we integrate over $B_r(x)$ instead of the boundary $\partial B_r(x)$.

Lemma 1.3. *Let $u : \Omega \to \mathbb{R}$ be a C^2-function and suppose $\overline{B}_r(x) \subseteq \Omega$. Then:*

$$\frac{d}{dr}\left(r^{-n} \int_{B_r(x)} u(y)d^n y \right) = r^{-n}\frac{1}{r} \int_0^r s \int_{B_s(x)} (\Delta u)(y)d^n y ds.$$

Proof. Again, we compute:

$$\begin{aligned}
\frac{d}{dr}\left(r^{-n}\int_{B_r(x)} u(y)d^n y\right) &= \frac{d}{dr}\int_{B_1(0)} u(x+rz)d^n z \text{ via } y = x+rz \\
&= \int_0^1 \int_{\partial B_s(0)} (\nabla u)(x+rz)\cdot z\mathcal{H}^{n-1}(dz)ds \\
&= \int_0^1 s\int_{\partial B_s(0)} (\nabla u)(x+rz)\cdot \frac{z}{s}\mathcal{H}^{n-1}(dz)ds \\
&= \int_0^1 s\int_{B_s(0)} r(\Delta u)(x+rz)d^n z ds \text{ by divergence theorem} \\
&= \frac{1}{r}\int_0^r a\int_{B_{a/r}(0)} (\Delta u)(x+rz)d^n z da \text{ via } a = rs \\
&= r^{-n}\frac{1}{r}\int_0^r a\int_{B_a(x)} (\Delta u)(y)d^n y da \text{ via } y = x+rz
\end{aligned}$$

and our result follows from renaming a to s. □

We now obtain a corollary analogous to the above.

Corollary 1.4. *Let $u : \Omega \to \mathbb{R}$ be a C^2 function and $x \in \Omega$. Define:*

$$\psi(r) := \frac{1}{\mathcal{H}^n(B_r(x))}\int_{B_r(x)} u(y)d^n y.$$

If $\Delta u = 0$ (respectively $\Delta u \geq 0$ or $\Delta u \leq 0$) then $\psi(r)$ is constant (respectively monotonically increasing in r, or monotonically decreasing in r). In particular:

$$\begin{aligned}
&\textit{If } \Delta u = 0 \textit{ then } u(x) = \frac{1}{\mathcal{H}^n(B_1)}r^{-n}\int_{B_r(x)} u(y)d^n y \\
&\textit{If } \Delta u \geq 0 \textit{ then } u(x) \leq \frac{1}{\mathcal{H}^n(B_1)}r^{-n}\int_{B_r(x)} u(y)d^n y \\
&\textit{If } \Delta u \leq 0 \textit{ then } u(x) \geq \frac{1}{\mathcal{H}^n(B_1)}r^{-n}\int_{B_r(x)} u(y)d^n y.
\end{aligned}$$

Proof. The first part of the corollary follows, as with the previous corollary, explicitly from 1.3. So, all that remains is to prove that

$$\lim_{r\searrow 0}\psi(r) = u(x).$$

The proof for this is the exact same as the previous corollary, except with $\partial B_r(x)$ replaced with $B_r(x)$ and $\mathcal{H}^{n-1}(dy)$ replaced with $d^n y$. □

As a consequence of this, we obtain the **maximum principles** that Yau likes to use in his proofs.

Corollary 1.5. Maximum Principle V1
Let $u : \Omega \to \mathbb{R}$ be a C^2 function, Ω be connected, and suppose that $\Delta u \geq 0$ (respectively $\Delta u \leq 0$) on Ω. Further, suppose that there existed a $x_0 \in \Omega$ such that

$$|\sup_\Omega u| < \infty \text{ (respectively } |\inf_\Omega u| < \infty) \text{ and } u(x_0) = \sup_\Omega u \text{ (respectively } u(x_0) = \inf_\Omega u).$$

Then u is constant. In particular, if $\Delta u = 0$ on Ω then u cannot achieve its supremum or infimum in Ω.

Proof. We will begin with the case where $\Delta u \geq 0$. In this case, let's define

$$M := \sup_{\Omega} u = u(x_0) \text{ and } \Omega_M := \{y \in \Omega \,:\, u(y) = M\}.$$

Since u is continuous, Ω_M is closed as a subset of Ω and furthermore $x_0 \in \Omega_M$ so Ω_M is non-empty. Now, let $z \in \Omega_M$ be arbitrary and choose $r > 0$ such that $\overline{B}_r(z) \subseteq \Omega$. Notice that

$$\Delta(u - M) = \Delta u \geq 0 \text{ and } u(y) - M = u(y) - \sup_{\Omega} u \leq 0 \text{ for all } y \in \Omega$$

so that $u - M$ non-positive, and also subharmonic. By the mean value inequality 1.4 applied to $u - M$ instead of u, and using that $z \in \Omega_M$ hence $u(z) - M = 0$, we have:

$$0 = u(z) - M \leq \frac{1}{\mathcal{H}^n(B_r(z))} \int_{B_r(z)} (u(y) - M) d^n y \leq 0 \text{ since } u(y) - M \leq 0 \text{ for all } y.$$

Thus

$$0 \leq \int_{B_r(z)} (u(y) - M) d^n y \leq 0 \text{ hence } \int_{B_r(z)} (u(y) - M) d^n y = 0.$$

But since $u(y) - M \leq 0$ for all $y \in B_r(z)$ and u is continuous, this integral vanishing must imply that the integrand vanishes. Hence

$$u(y) - M = 0 \text{ for all } y \in B_r(z).$$

Thus, recalling the definition of Ω_M we have

$$B_r(z) \subseteq \Omega_M$$

and hence Ω_M is open in Ω. Thus Ω_M is clopen in Ω and, since Ω is connected, we have $\Omega = \Omega_M$. By the definition of Ω_M this implies:

$$u(y) = M \text{ for all } y \in \Omega, \text{ hence } u \text{ is constant, as desired.}$$

Let's now turn to the second part of the theorem: assuming that $\Delta u \leq 0$ and that there exists $x_0 \in \Omega$ such that

$$u(x_0) = \inf_{\Omega} u.$$

Define $v := -u$. Then $\Delta v = -\Delta u \geq 0$ and

$$v(x_0) = -u(x_0) = -\inf_{\Omega} u = \sup_{\Omega}(-u) = \sup_{\Omega} v.$$

Therefore v satisfied the assumptions of the first part of the theorem hence v is contant. But then $u = -v$ therefore u must be constant, as desired. $\square$

Corollary 1.6. Maximum Principle V2
Let $u : \overline{\Omega} \to \mathbb{R}$ be a continuous function such that $u|_{\Omega}$ is C^2. Assume that $\Delta u \geq 0$ (respectively $\Delta u \leq 0$) on Ω. Furthermore, assume that Ω is bounded and connected. Then:

$$\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u \quad \textit{(respectively } \inf_{\overline{\Omega}} u = \inf_{\partial\Omega} u).$$

Proof. For the first part of the corollary, we assume $\Delta u \geq 0$. Since $\overline{\Omega}$ is compact and u is continuous on $\overline{\Omega}$ it follows that $|\sup_{\overline{\Omega}} u| < \infty$ and there exists a $x_0 \in \overline{\Omega}$ such that

$$u(x_0) = \sup_{\overline{\Omega}} u.$$

If $x_0 \in \partial\Omega$ then we are done since then $\partial\Omega \subseteq \overline{\Omega}$ implies

$$u(x_0) \leq \sup_{\partial\Omega} u \leq \sup_{\overline{\Omega}} u = u(x_0)$$

hence

$$\sup_{\partial\Omega} u = u(x_0) = \sup_{\overline{\Omega}}.$$

On the other hand, suppose $x_0 \notin \partial\Omega$ and hence $x_0 \in \Omega$. Then, by 1.5 it follows that u is constant on Ω. But now, since u is continuous on $\overline{\Omega}$ it follows that u is constant on $\overline{\Omega}$ as well. Indeed, let $M := u(x_0)$ and let $y \in \overline{\Omega}$ be arbitrary. If $y \in \Omega$ then $u(y) = M$ since $u|_\Omega$ is constant so let's assume instead that $y \in \partial\Omega$. Then, by definition, there exists a sequence $y_1, y_2, ...$ in Ω such that

$$y = \lim_{k\to\infty} y_k.$$

Since u is continuous on $\overline{\Omega}$ we then have

$$u(y) = \lim_{k\to\infty} u(y_k) = \lim_{k\to\infty} M = M$$

hence indeed u is constant (and equal to M) on all of $\overline{\Omega}$. But now:

$$\sup_{\partial\Omega} u = \sup_{\partial\Omega} M = M = \sup_{\overline{\Omega}} u$$

as desired.

For the case where $\Delta u \leq 0$ we do the same trick as in the proof of 1.5 where we let $v := -u$ and notice that $\Delta v \geq 0$. □

As a final corollary, we demonstrate how one can obtain the Harnack inequalities from the above. There are two versions of these inequalities. The first version is one we can prove directly and this is done below.

Theorem 1.7. Harnack Inequality V1
Let $u : \Omega \to \mathbb{R}$ be a C^2 function with $\Delta u = 0$ and let $U \subseteq \Omega$ be a connected bounded open subset with $\overline{U} \subseteq \Omega$ and $\overline{U}$ path connected. Then there exists a constant $C = C(n, \Omega, U) > 0$ (this notation means that C depends only on n, Ω and U, but not on u) such that if $u \geq 0$ on Ω then

$$\sup_{\overline{U}} u \leq C \inf_{\overline{U}} u.$$

Proof. We begin by considering the case where $y \in \Omega$, $\overline{B}_{4r}(y) \subseteq \Omega$ and $U = B_r(y)$. Towards this end, let $x_1, x_2 \in B_r(y)$ be arbitrary and notice that for any $z \in B_r(x_1)$ we have

$$|z - y| \leq |z - x_i| + |x_i - y| < r + r = 2r$$

hence

$$B_r(x_1) \subseteq B_{2r}(y).$$

Similarly, for any $z \in B_{2r}(y)$ we have:

$$|z - x_2| \leq |z - y| + |y - x_2| < 2r + r = 3r$$

therefore

$$B_{2r}(y) \subseteq B_{3r}(x_2).$$

Then, making use of the mean value equality of 1.4 for $\Delta u = 0$ we have

$$\begin{aligned} u(x_1) &= \frac{1}{\mathcal{H}^n(B_r(x_1))} \int_{B_r(x_1)} u(z) d^n z \\ &\leq \frac{1}{\mathcal{H}^n(B_r(x_1))} \int_{B_{2r}(y)} u(z) d^n z \text{ since } u \geq 0 \text{ and } B_r(x_1) \subseteq B_{2r}(y) \end{aligned}$$

and

$$\begin{aligned} u(x_2) &= \frac{1}{\mathcal{H}^n(B_{3r}(x_2))} \int_{B_{3r}(x_2)} u(z) d^n z \\ &\geq \frac{1}{\mathcal{H}^n(B_{3r}(x_2))} \int_{B_{2r}(y)} u(z) d^n z \text{ since } u \geq \text{ and } B_{2r}(y) \subseteq B_{3r}(x_2). \end{aligned}$$

Combining these two inequalities we have

$$\begin{aligned} u(x_1) &\leq \frac{1}{\mathcal{H}^n(B_r(x_1))} \int_{B_{2r}(y)} u(z) d^n z \\ &= \frac{\mathcal{H}^n(B_{3r}(x_2))}{\mathcal{H}^n(B_r(x_1))} \frac{1}{\mathcal{H}^n(B_{3r}(x_2))} \int_{B_{2r}(y)} u(z) d^n z \\ &\leq \frac{\mathcal{H}^n(B_{3r}(x_2))}{\mathcal{H}^n(B_r(x_1))} u(x_2) \\ &= 3^n u(x_2). \end{aligned}$$

So, in summary we've shown that if $B_{4r}(y) \subseteq \Omega$ then for all $x_1, x_2 \in B_r(y)$ we have

$$u(x_1) \leq 3^n u(x_2)$$

and hence for every $x_2 \in B_r(y)$ we have

$$\sup_{B_r(y)} u \leq 3^n u(x_2)$$

therefore finally:

$$\sup_{B_r(y)} u \leq 3^n \inf_{B_r(y)} u.$$

Next, we consider the general case of U bounded open and connected with $\overline{U} \subseteq \Omega$. Since u is continuous and $\overline{U}$ is compact, there exists $x_1, x_2 \in \overline{U}$ such that

$$\sup_{\overline{U}} u = u(x_1) \text{ and } \inf_{\overline{U}} u = u(x_2).$$

Since $\overline{U}$ is path-connected, it follows that there exists a continuous map

$$\gamma : [0, 1] \to \overline{U} \text{ such that } \gamma(0) = x_1 \text{ and } \gamma(1) = x_2.$$

Since $\overline{U}$ is compact and $\overline{U} \subseteq \Omega$ with Ω open, it follows that

$$\text{dist}(\overline{U}, \Omega^c) \in \mathbb{R}_{>0} \cup \{+\infty\}$$

and so we can choose an $r > 0$ such that

$$5r < \text{dist}(\overline{U}, \Omega^c).$$

Then, for every $y \in \gamma([0,1]) \subseteq \overline{U}$ we have $\overline{B}_{4r}(y) \subseteq \Omega$. Let:

$$\overline{B}_r(\overline{U}) := \{y \in \mathbb{R}^n \ : \ \mathrm{dist}(\{y\}, \overline{U}) \leq r\} \subseteq \Omega$$

and

$N :=$ the maximum number of disjoint balls of radius $r/5$ that can be contained in the compact set $\overline{B}_r(\overline{U})$.

Since r depended only on U and Ω, it follows that N depends only on U, Ω, n. Notice that N is finite since if $B_{r/5}(y_1), ..., B_{r/5}(y_m)$ are disjoint balls contained in $\overline{B}_r(\overline{U})$ then

$$\mathcal{H}^n(\overline{B}_r(\overline{U})) \geq \sum_{k=1}^{m} \mathcal{H}^n(B_{r/5}(y_k)) = \mathcal{H}^n(B_1)(r/5)^n k$$

so that

$$k \leq \frac{\mathcal{H}^n(\overline{B}_{r/5}(\overline{U}))}{\mathcal{H}^n(B_{r/5})} < \infty \text{ since } \overline{B}_{r/5}(\overline{U}) \text{ is compact.}$$

Since N is the supremum over all such k's we indeed have that N is bounded by the right hand side of the above inequality. Now,

$$\gamma([0,1]) \subseteq \bigcup_{y \in \gamma([0,1])} B_{r/5}(y)$$

and $\gamma([0,1])$ is compact therefore there exists a finite subcover

$$\gamma([0,1]) \subseteq \bigcup_{k=1}^{m} B_{r/5}(y_k) \ \text{ for some } y_1, ..., y_k \in \gamma([0,1]).$$

By the Vitali covering theorem, there exists a subset $\{z_1, ..., z_\ell\} \subseteq \{y_1, ..., y_m\}$ such that

$$B_{r/5}(z_i) \cap B_{r/5}(z_j) = \emptyset \ \text{ for } i \neq j \text{ and}$$

$$\gamma([0,1]) \subseteq \bigcup_{k=1}^{\ell} B_r(z_k).$$

Furthermore, by our definition of N we have

$$\ell \leq N.$$

Reordering the z_i's so that:

$$x_0 = \gamma(0) \in B_r(z_1), \ x_1 = \gamma(1) \in B_r(z_\ell), \ \text{ and } \ B_r(z_i) \cap B_r(z_{i+1}) \neq \emptyset \ \text{ for all } i$$

we can arbitrarily choose $z_i' \in B_r(z_i) \cap B_r(z_{i+1})$ for each $i = 1, ..., \ell - 1$ to obtain, from the first part of our proof, that

$$u(x_0) \leq 3^n u(z_1') \leq 3^{2n} u(z_2') \leq \cdot \leq 3^{\ell n} u(z_\ell') \leq 3^{\ell n+1} u(x_1).$$

Since $\ell \leq N$ we arrive at

$$u(x_0) \leq 3^{Nn+1} u(x_1).$$

But since we chose x_0, x_1 such that

$$u(x_0) = \sup_{\overline{U}} u \ \text{ and } \ u(x_1) = \inf_{\overline{U}} u$$

we arrive at

$$\sup_{\overline{U}} u \leq 3^{Nn+1} \inf_{\overline{U}} u$$

as desired. □

The key to the second version of Harnack's inequality is the following.

Lemma 1.8. *Let $u : \Omega \to \mathbb{R}$ be a C^3 function and suppose $\Delta u = 0$. Then $v := |\nabla u|^2$ is subharmonic (i.e. $\Delta v \geq 0$). More generally, if u is any C^3 function (not necessarily harmonic) we have*

$$\Delta|\nabla u|^2 = 2|\operatorname{Hess}(u)|^2 + 2\langle \nabla u, \nabla(\Delta u)\rangle$$

where $|A|^2$ is defined to be the "L^2-matrix norm" of A, defined during the proof, and $\langle -, - \rangle$ is the dot product.

Proof. Here we will let $\operatorname{Hess}(u)$ denote the Hessian matrix of u:

$$\operatorname{Hess}(u) = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix}.$$

Furthermore, given an $n \times n$ matrix A with entries $A_{i,j}$ we write:

$$|A|^2 := \sum_{i,j} A_{i,j}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j}^2 \geq 0.$$

Let's now compute:

$$\begin{aligned}
\Delta|\nabla u|^2 &= \Delta\left(\sum_{i=1}^{n} (\partial_i u)^2\right) \\
&= \sum_{j=1}^{n} \partial_j^2 \left(\sum_{i=1}^{n} (\partial_i u)^2\right) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \partial_j^2 \left((\partial_i u)^2\right) \\
&= \sum_{i,j} \partial_j \left(2(\partial_i u)\partial_j \partial_i u\right) \quad \text{by product rule for differentiation} \\
&= \sum_{i,j} \left(2(\partial_j \partial_i u)^2 + 2(\partial_i u)(\partial_j^2 \partial_i u)\right) \quad \text{by product rule again} \\
&= 2\sum_{i,j} (\partial_j \partial_i u)^2 + 2\sum_{i,j} (\partial_i u)(\partial_i \partial_j^2 u) \\
&= 2|\operatorname{Hess}(u)|^2 + 2\sum_{i} (\partial_i u)(\partial_i \Delta u) \\
&= 2|\operatorname{Hess}(u)|^2 + 2\langle \nabla u, \nabla(\Delta u)\rangle
\end{aligned}$$

where $\langle \vec{x}, \vec{y} \rangle$ denotes the dot product of vectors. Since $\Delta u = 0$ by assumption we have

$$\Delta|\nabla v|^2 = 2|\operatorname{Hess}(u)|^2 \geq 0$$

as desired. $\square$

This second version of Harnack's inequality is actually with regards to the heat equation

$$\partial_t u - \Delta u = 0$$

and, in the more general case where we are on a compact Riemannian manifold with non-negative Ricci curvature, is often called the **Li-Yau Harnack inequality**.

Theorem 1.9. Harnack Inequality V2
Let $u : [T_-, T_+] \times \overline{\Omega} \to \mathbb{R}$ be a C^4-function which solves $\partial_t u - \Delta u = 0$ on some open neighborhood of $[T_-, T_+] \times \overline{\Omega} \subseteq \mathbb{R}^{n+1}$ and satisfies $u(t, x) > 0$ for all $(t, x) \in [T_-, T_+] \times \overline{\Omega}$. Furthermore, assume that on $(T_-, T_+] \times \partial\Omega$ we have

$$\frac{1}{u}\partial_t u - \frac{1}{u^2}|\nabla u|^2 \geq -\frac{n^2/2}{t - T_-}.$$

Then the above inequality actually holds on all of $(T_-, T_+] \times \overline{\Omega}$.

Proof. Throughout, when we write $\mathrm{Hess}(f)$ for a function $f : [T_-, T_+] \times \overline{\Omega} \to \mathbb{R}$ we **only** consider the derivatives in the $\overline{\Omega}$-directions (not the t-directions) so that $\mathrm{Hess}(f)$ is still an $n \times n$ matrix. Similarly, ∇f does **not** include the ∂_t-direction so that it is still a vector of length n.

Now, u is continuous and $[T_-, T_+] \times \overline{\Omega}$ is compact therefore there exists $(t_0, x_0) \in [T_-, T_+] \times \overline{\Omega}$ so that

$$\inf_{[T_-,T_+]\times\overline{\Omega}} u = u(t_0, x_0) > 0$$

and therefore the function $f := \log(u)$ is smooth on $[T_-, T_+] \times \overline{\Omega}$. We compute, using the chain rule, that:

$$\begin{aligned}
\nabla f &= \frac{1}{u}\nabla u \\
\text{so } \Delta f &= \sum_{j=1}^{n} \partial_j^2 \log(u) \\
&= \sum_{j=1}^{n} \partial_j \left(\frac{1}{u}\partial_j u\right) \\
&= \sum_{j=1}^{n} \left(-\frac{1}{u^2}(\partial_j u)(\partial_j u) + \frac{1}{u}\partial_j^2 u\right) \text{ by product rule} \\
&= -\frac{1}{u^2}|\nabla u|^2 + \frac{1}{u}\Delta u \\
&= -|\nabla f|^2 + \frac{1}{u}\Delta u \\
\text{and thus } \partial_t \Delta f &= \Delta \partial_t f \\
&= \Delta\left(\frac{1}{u}\partial_t u\right) \\
&= \Delta\left(\frac{1}{u}\Delta u\right) \\
&= \Delta(\Delta f + |\nabla f|^2) \\
&= \Delta(\Delta f) + 2|\mathrm{Hess}(f)|^2 + 2\langle \nabla f, \nabla(\Delta f)\rangle \text{ by lemma 1.8.}
\end{aligned}$$

Now, for an $n \times n$ matrix A with entries A_{ij} we can notice that our $|A|^2$ defined earlier is equal to the result you get if you interpret A as a vector of length n^2 and consider the dot product of A with itself. Interpreting the $n \times n$ identity matrix as a vector of length n^2 in the same way, we can write the trace of A as a dot product via:

$$\mathrm{Tr}(A) = \langle A, I\rangle.$$

Applying the Cauchy-Schwarz inequality we get:

$$|\mathrm{Tr}(A)|^2 \leq |A|^2|I|^2 = n^2|A|^2.$$

Applying this to $A = \mathrm{Hess}(f)$ and noticing that Δf is the trace of $\mathrm{Hess}(f)$ gives us:

$$\frac{1}{n^2}(\Delta f)^2 \leq |\,\mathrm{Hess}(f)|^2.$$

So if we let $F := \Delta f$ then F satisfies the differential inequality:

$$\partial_t F - \Delta F - 2\langle \nabla f, \nabla F\rangle - \frac{2}{n^2}F^2 \geq 0.$$

Now, consider instead the function:

$$v(t,x) := -\frac{n^2/2}{t - T_-} \text{ on } (T_-, T_+] \times \overline{\Omega}.$$

Since our gradients ∇ don't include the t-directions we have $\nabla v = 0$. Furthermore:

$$\partial_t v = \frac{n^2/2}{(t-T_-)^2} = \frac{2}{n^2}v^2 = \frac{2}{n^2}\frac{n^4/4}{(t-T_-)^2}.$$

Thus v satisfies the differential equality

$$\partial_t v - \Delta v - 2\langle \nabla f, \nabla v\rangle - \frac{2}{n^2}v^2 = 0.$$

Now, F is smooth on the compact set $[T_-, T_+] \times \overline{\Omega}$ and is therefore bounded below. Since

$$\lim_{t \searrow T_-} v(t,x) = -\infty \text{ for all x}$$

it follows that there exists a $0 < \delta < T_+ - T_-$ such that:

$$v(t,x) \leq F(t,x) \text{ for all } (t,x) \in (T_-, T_- + \delta] \times \overline{\Omega}.$$

Now, suppose for contradiction that there existed $(t,x) \in (T_- + \delta, T_+] \times \overline{\Omega}$ such that

$$h(t,x) := F(t,x) - v(t,x) < 0.$$

Let $A \in \mathbb{R}$, to be determined later, and

$$-\epsilon := \inf_{(t,x)\in[T_-+\delta, T_+]\times\overline{\Omega}} e^{A(t-T_--\delta)}h(t,x) < 0$$

and

$$S := \{s \in [T_- + \delta, T_+] \;:\; \exists x \in \overline{\Omega} \text{ such that } e^{A(s-T_--\delta)}h(s,x) = -\epsilon\}.$$

Since $S \neq \emptyset$ by assumption, h is continuous, and since $h(T_- + \delta, -) \geq 0$ it follows that

$$t_0 := \inf S \in (T_- + \delta, T_+]$$

and that there exists $x_0 \in \overline{\Omega}$ such that

$$e^{A(t_0-T_--\delta)}h(t_0, x_0) = -\epsilon.$$

By our assumption concerning u on $(T_-, T_+] \times \partial\Omega$ it follows that $h \geq 0$ on $[T_- + \delta, T_+] \times \partial\Omega$ and therefore $x_0 \in \Omega$. Since Ω is open is follows that x_0 is a local minimum for $h(t_0, -)$ and hence:

$$\nabla h(t_0, x_0) = 0 \text{ and } \Delta h(t_0, x_0) \geq 0.$$

Furthermore, since $t_0 = \inf S$ we have

$$\partial_t \left(e^{A(t-T_- -\delta)} h(t,x) \right) \Big|_{(t_0,x_0)} \le 0$$

and so

$$\partial_t h(t_0,x_0) \le -Ah(t_0,x_0).$$

Putting this together gives us:

$$\partial_t h(t_0,x_0) - \Delta h(t_0,x_0) - 2\langle \nabla f(t_0,x_0), \nabla h(t_0,x_0)\rangle \le -Ah(t_0,x_0).$$

Inserting into this equation our definition of $h = F - v$ and using the equations satisfied by F and v give us:

$$\begin{aligned}
\frac{2}{n^2}F(t_0,x_0)^2 &\le \partial_t F(t_0,x_0) - \Delta F(t_0,x_0) - 2\langle \nabla f(t_0,x_0), \nabla F(t_0,x_0)\rangle \\
&\le \partial_t v(t_0,x_0) - \Delta v(t_0,x_0) - 2\langle \nabla f(t_0,x_0), \nabla v(t_0,x_0)\rangle - AF(t_0,x_0) + Av(t_0,x_0) \\
&= \frac{2}{n^2} v(t_0,x_0)^2 - A(F(t_0,x_0) - v(t_0,x_0)).
\end{aligned}$$

It's now the time to choose A, but we should remark that we are not allowed to make A dependent on t_0 or x_0 since we used A in the definition of t_0 and x_0. Since $F(t_0,x_0) - v(t_0,x_0) < 0$ by assumption, rearranging the above inequality leads us to

$$F(t_0,x_0) + v(t_0,x_0) \ge -\frac{An^2}{2}.$$

But now consider

$$B := \sup_{[T_-+\delta, T_+]\times\overline{\Omega}} (F+v).$$

This supremum is finite since our functions are smooth on the compact domain $[T_- + \delta, T_+] \times \overline{\Omega}$. If we then choose any $A \in \mathbb{R}$ with $A < -2B/n^2$ we obtain a contradiction. Therefore we actually have

$$-\frac{n^2/2}{t-T_-} \le F(t,x) \ \text{ for all } (t,x) \in [T_-,T_+]\times\overline{\Omega}.$$

Recalling now that

$$F = \Delta f = \frac{1}{u}\Delta u - \frac{|\nabla u|^2}{u^2}$$

and that $\Delta u = \partial_t u$ we arrive at our desired result. □

A good question to ask is: why is the above theorem called a "Harnack Inequality"? How is it related to the previous Harnack Inequality? The following corollary is the reason for this name.

Corollary 1.10. *Let $u : [T_-,T_+]\times\overline{\Omega} \to \mathbb{R}_{>0}$ be a C^4 function solving $\partial_t u - \Delta u = 0$ on a neighborhood of $[T_-,T_+]\times\overline{\Omega}$ and satisfying*

$$\frac{1}{u}\partial_t u - \frac{1}{u^2}|\nabla u|^2 \ge -\frac{n^2/2}{t-T_-} \ \text{ on } \ (T_-,T_+]\times\partial\Omega.$$

Then, for every $(x_1,t_1),(x_2,t_2) \in [T_-,T_+]\times\overline{\Omega}$ with $t_1 < t_2$ we have:

$$u(t_1,x_1) \le \left(\frac{t_2-T_-}{t_1-T_-}\right)^{n/2} \exp\left(|x_2-x_1|^2/4(t_2-t_1)\right) u(t_2,x_2).$$

Therefore, whenever $t_1 < t_2$ we have

$$\sup_{\overline{\Omega}} u(t_1, -) \leq \left(\frac{t_2 - T_-}{t_1 - T_-}\right)^{n/2} \exp\left(\operatorname{diam}(\overline{\Omega})^2/4(t_2 - t_1)\right) \inf_{\overline{\Omega}} u(t_2, -)$$

where

$$\operatorname{diam}(\overline{\Omega}) := \sup_{x_1, x_2 \in \overline{\Omega}} |x_2 - x_1|.$$

2 Harmonic functions etc. on a Manifold

Suppose (M, g) is an n-dimensional Riemannian manifold. Recall that the Laplacian (on functions) on M is given in local coordinates by

$$\Delta_g f = \frac{1}{\sqrt{|\det(g)|}} \sum_{ij} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{|\det(g)|} \frac{\partial f}{\partial x^j} \right).$$

The manifold (M, g) is also equipped with a measure μ_g which in local coordinates is of the form

$$\mu_g(dx) = \sqrt{|\det(g)|}\mathcal{H}^n(dx) \text{ with } \mathcal{H}^n \text{ equal to the Lebesgue measure in that chart.}$$

Below, we give the local-coordinate expressions for (j'th component of) the gradient of a function, and the divergence of a vector field on M:

$$(\nabla f)^j = \sum_i g^{ij} \frac{\partial f}{\partial x^i}$$
$$\operatorname{div}(X) = \frac{1}{\sqrt{|\det(g)|}} \sum_i \frac{\partial}{\partial x^i} \left(\sqrt{|\det(g)|} X^i \right).$$

Furthermore, we introduce the convention that if X is a vector field on M then we denote by:

$$|X| := \sqrt{\sum_{ij} g_{ij} X^i X^j} = \sqrt{g(X, X)}$$

the scalar function whose value at $p \in M$ is the length (with respect to the inner product $g(p)$) of the tangent vector $X(p) \in T_p M$.

Definition 2.1. Let $\operatorname{diag}(M) := \{(x, x) \in M \times M \ : \ x \in M\}$. A **Green's function** on M for the Laplacian Δ_g is a smooth function

$$G : (M \times M) \setminus \operatorname{diag}(M) \to \mathbb{R}$$

with the following properties:

1. For each $y \in M$, the function $G_y(x) := G(x, y)$ is in $L^1_{loc}(M, d\mu_g)$.
2. For each $x, y \in M$ with $x \neq y$ we have $\Delta G_y(x) = 0$.
3. For each $y \in M$, the function $G_y : M \setminus \{y\} \to \mathbb{R}$ satisfies $G_y(x) > 0$ for all $x \neq y$,
$$\lim_{x \to y} G_y(x) = \infty \text{ and } \{x \in M \ : \ G_y(x) \geq a\} \text{ is compact for all } a > 0.$$
4. $\nabla G_y(x) \neq 0$ for all $x \neq y$, $x, y \in M$.

As a good example, notice that $G(x,y) = |x-y|^{2-n}$ is a Green's function on $M = \mathbb{R}^n$ (recall: we assume throughout that $n \geq 3$).

Lemma 2.2. *For $p \in M$, denote*

$$b_p(x) := \begin{cases} G_p(x)^{-1/(n-2)} & \text{if } x \neq p \\ 0 & \text{if } x = p. \end{cases}$$

Then $b_p(x)$ is smooth away from $x = p$, it is continuous at $x = p$, and it solves

$$\Delta_g b_p = \frac{n-1}{b_p} |\nabla b_p|^2 \ \text{ away from } x = p.$$

Furthermore, the sub-level-sets $\{x \in M \ : \ b_p(x) \leq a\}$ are smooth n-dimensional compact submanifolds of M with boundary $b^{-1}(\{a\})$.

Now, the tangent space to the level set $b^{-1}(\{a\})$ at a point x is the subspace of $T_x M$ given by

$$T_x b^{-1}(\{a\}) = \{\vec{v} \in T_x M \ : \ g_x(\vec{v}, \nabla b_p(x)) = 0\} \subseteq T_x M.$$

By restricting the inner product g_p on $T_x M$ to $T_x b^{-1}(\{a\})$ we get an inner product h on $T_x b^{-1}(\{a\})$. Doing this for each $x \in b^{-1}(\{a\})$ gives us a Riemannian metrix h on $b^{-1}(\{a\})$. Let

$$A_g := \text{ volume measure of the Riemannian manifold } (b^{-1}(\{a\}), h).$$

Lemma 2.3. *Let $f : M \to \mathbb{R}$ be a smooth function. Then*

$$\int_{b^{-1}(\{a\})} f(x) |\nabla b|(x) A_g(dx) = \int_{\{b \leq a\}} \operatorname{div}(f \nabla g)(x) \mu_g(dx).$$

Proposition 2.4. *Let $f : M \to \mathbb{R}$ be a smooth function. Then*

$$\frac{d}{dr} \left(r^{1-n} \int_{b^{-1}(\{a\})} f(x) |\nabla b|(x) A_g(dx) \right) = r^{1-n} \int_{\{b \leq a\}} (\Delta_g f)(x) \mu_g(dx).$$

We can now use this to develop mean value inequalities, maximum principles and Harnack inequalities for the Laplacian on (M, g)! There are two more important ingredients, however:

1. We will need to compare the volume of the level set $b^{-1}(\{a\})$ to the volume of the sphere $\partial B_r(p)$, and then compare the volume of $\partial B_r(p)$ **in** M to r^{1-n}.

2. When doing Harnack inequalities, we will need to contend with the fact that our computation of $\Delta |\nabla u|^2$ on $\mathbb{R}^n$ doesn't exactly yield the same result on a manifold (M, g). There is an extra error term involving Ricci curvature.

3 The Heat Equation on $\mathbb{R}^n$

Similar to how we needed to use level sets of a Green's function to study the Laplacian on a manifold, we will need to use level sets of a fundamental solution to study the heat equation. However, for the heat equation we are forced to do this already on $\mathbb{R}^n$!

The heat equation is given by

$$\begin{cases} \partial_t u - \Delta u &= 0 \\ u(x, 0) &= u_0(x) \end{cases}$$

where u_0 is a given function of x. Assuming, for example, that $u_0 \in L^2(\mathbb{R}^n)$ we can solve the heat equation by introducing the function:

$$K(x,t) := (4\pi t)^{-n/2} \exp(-|x|^2/4t).$$

It then follows that the solution u to the heat equation is given by

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t) u_0(y) d^n y.$$

The analogues of our balls $B_r(x)$ from our analysis of Laplace's equation here will be the subsets:

$$E_r(x,t) := \{(y,s) \in \mathbb{R}^n \times \mathbb{R} \ : \ s < t \text{ and } K(x-y,t-s) \geq r^{-n}\}.$$

Let's think about what $E_r(x,t)$ looks like for a bit. If we hold $s = s_0$ fixed we can compute that

$$E_r(x,t) \cap \{(y,s) \ : \ s = s_0\} = \begin{cases} \emptyset & \text{if } s \geq t \text{ or } s < t - \frac{r^2}{4\pi} \\ \{(y,s_0) \ : \ |y|^2 \leq \frac{n}{2\pi} 4\pi(t-s) \log(r^2/4\pi(t-s))\} & \text{otherwise.} \end{cases}$$

So, for s in the range $t - \frac{r^2}{4\pi} \leq s < t$, the y-cross-section of $E_r(x,t)$ looks like a ball of radius:

$$\sqrt{\frac{n}{2\pi} 4\pi(t-s) \log(r^2/4\pi(t-s))}.$$

To get some intuition for this, you can write it as

$$r\sqrt{n\Phi(4\pi(t-s)/r^2)/2\pi} \ \text{ where } \ \Phi(a) = a\log(1/a)$$

and you can view a graph of $\Phi(a)$ on Wolfram.

Lemma 3.1.

$$\int_{E_r(x,t)} \frac{|x-y|^2}{(t-s)^2} d^n y ds = 4r^n.$$

TODO: The function $|x-y|^2/(t-s)^2$ should arise from viewing this as an integral in $\mathbb{R}^n \times \mathbb{R}_+$ with respect to some sort of hyperbolic metric.

Proposition 3.2. *If $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$ is an open subset and $u : \Omega \to \mathbb{R}$ is C^2 then for $\overline{E_r(x,t)} \subseteq \Omega$ we have*

$$\frac{d}{dr}\left(r^{-n} \int_{E_r(x,t)} u(y,s) \frac{|x-y|^2}{(t-s)^2} d^n y ds\right) = -\frac{4n}{r^{n+1}} \int_{E_r(x,t)} (\partial_s u - \Delta u) \log(r^n K(x-y,t-s)) d^n y ds.$$

Corollary 3.3. *Let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$ be open, $u : \Omega \to \mathbb{R}$ a C^2 function and denote*

$$\varphi(r) := r^{-n} \int_{E_r(x,t)} u(y,s) \frac{|x-y|^2}{(t-s)^2} d^n y ds.$$

If $\partial_t u - \Delta u = 0$ then $\varphi(r)$ is constant. If $\partial_t u - \Delta u \geq 0$ (i.e. u is a supersolution) then $\varphi(r)$ is monotonically increasing for $r \leq 1$. If $\partial_t u - \Delta u \leq 0$ (so u is a subsolution) then $\varphi(r)$ is monotonically decreasing for $r \leq 1$. In particular:

$$\text{If } \partial_t u - \Delta u = 0 \text{ then } u(x,t) = \frac{1}{4r^n} \int_{E_r(x,t)} u(y,s) \frac{|x-y|^2}{(t-s)^2} d^n y ds$$

$$\text{If } \partial_t u - \Delta u \geq 0 \text{ then } u(x,t) \leq \frac{1}{4r^n} \int_{E_r(x,t)} u(y,s) \frac{|x-y|^2}{(t-s)^2} d^n y ds$$

$$\text{If } \partial_t u - \Delta u \leq 0 \text{ then } u(x,t) \geq \frac{1}{4r^n} \int_{E_r(x,t)} u(y,s) \frac{|x-y|^2}{(t-s)^2} d^n y ds.$$

4 Existence of Green's Functions on a Manifold

We mostly follow Aubin's books and a few papers here.

Theorem 4.1. *Let $(\overline{\Omega}, g)$ be a compact oriented n-dimensional Riemannian manifold ($n \geq 2$) with boundary $\partial\Omega$. Then there exists a function*

$$G : \overline{\Omega} \times \overline{\Omega} \setminus \operatorname{diag}(\overline{\Omega}) \to \mathbb{R}$$

satisfying the following:

1. *G is smooth. Furthermore, for all $x, y \in \overline{\Omega}$ with $x \neq y$ we have $G(x,y) = G(y,x)$. If $x, y \in \Omega$ with $x \neq y$ then we also have $G(x,y) > 0$. Finally, if $x \in \overline{\Omega}$ and $y \in \partial\Omega$ with $x \neq y$ then $G(x,y) = 0$.*
2. *Fix $x \in \Omega$. Then there exists $C = C(d_g(x, \partial\Omega)) > 0$ such that for all $y \in \overline{\Omega}$ with $y \neq x$ we have:*
   - $|G(x,y)| < \begin{cases} Cd_g(x,y)^{2-n} & \text{if } n \geq 3 \\ C(1 + |\log(d_g(x,y))|) & \text{if } n = 2, \end{cases}$
   - $|\nabla_y G(x,y)| < Cd_g(x,y)^{1-n}$,
   - $|\nabla_y^2 G(x,y)| < Cd_g(x,y)^{-n}$.
3. *For all $\phi \in C^2(\overline{\Omega})$ we have:*

$$\phi(x) = \int_\Omega G(x,y)(\Delta\phi)(y)\mu_g(dy) - \int_{\partial\Omega} \langle \widehat{n}, \nabla_y G(x,y)\rangle \phi(y) A_g(dy)$$

   for $\widehat{n}$ the outward normal of $\partial\Omega$ and $A_g(dy)$ the surface measure on $\partial\Omega$.

Proof. **(Sketch)** Fix $x \in \Omega$ and define

$$H(x,y) := \begin{cases} ((n-2)\omega_{n-2})^{-1} d_g(x,y)^{2-n} f(d_g(x,y)) & \text{for } n \geq 3 \\ -(2\pi)^{-1} f(d_g(x,y)) \log(d_g(x,y)) & \text{for } n = 2 \end{cases}$$

where $f(r) = 0$ for $r > (k+1)^{-1} \operatorname{inj}_g(x)$ with $k \in \mathbb{N}$, $k > n/2$ to be determined. We then iteratively define:

- $\Gamma(x,y) := \Gamma_1(x,y) := -\Delta_y H(x,y)$,
- $\Gamma_{i+1}(x,y) := \int_\Omega \Gamma_i(x,z)\Gamma_1(z,y)\mu_g(dz)$.

Then, for our $k > n/2$ we let F be the solution to

$$\begin{cases} \Delta_y F(x,y) & = \Gamma_{k+1}(x,y) \\ F(x,y) & = 0 \text{ for } y \in \partial\Omega. \end{cases}$$

Using all of this we can finally define G as:

$$G(x,y) := H(x,y) + \sum_{i=1}^{k} \int_\Omega \Gamma_i(x,z) H(z,y) \mu_g(dz) + F(x,y)$$

and this will work. □

We now use the above result for the case of (M^n, g) complete and non-compact. Fix $x \in M$. By Nash's theorem (this is probably overkill) there exists a sequence of compact submanifolds $\overline{\Omega_i} \subseteq M$ with smooth boundary such that

$$x \in \Omega_1, \ \overline{\Omega}_i \subseteq \Omega_{i+1}, \ \text{ and } \ M = \bigcup_{i=1}^{\infty} \Omega_i.$$

Let $G_i(x,y)$ denote the (Dirichlet) Green's function for Ω_i from the above theorem.

Consider then the sequence of functions $G_i(x,-)$. By the maximum principle, Dirichlet boundary conditions, and positivity of the Green's functions it follows that

$$G_{i+1}(x,-) \geq G_i(x,-) \text{ on } \Omega_i \text{ for all } i.$$

Furthermore, for every compact subset $K \subseteq M \setminus \{x\}$ one can use the maximum principle and the Harnack inequality to prove that the oscillations

$$\operatorname{osc}_K G_i(x,-) := \sup\{G_i(x,y) \,:\, y \in K\} - \inf\{G_i(x,y) \,:\, y \in K\}$$

are uniformly bounded in i for i sufficiently large such that $K \subseteq \Omega_i$. So, in conclusion, to define G as a limit of the G_i's we only need some sort of uniform boundedness of the sequence.

If there existed an $f \in C^\infty(M)$ such that $f(y) > 0$ for all $y \in M$, f is non-constant, and $\Delta f = 0$ then we can use f to demonstrate sufficient boundedness for us to define

$$G(x,-) := \lim_{i\to\infty} G_i(x,-)$$

and the resulting Green's function $G(x,y)$ will be **positive** and symmetric.

Otherwise, no such f exists. This implies the following:

if u is a bounded harmonic function on (M,g) then u is constant.

We then claim that there exists $a_i \in \mathbb{R}$ and a subsequence of $f_i(x,-) := G_i(x,-) - a_i$ that converges on compact subsets of $M \setminus \{x\}$. The difficulty then is the potential for a_i to depend on x. This is what we use the above consequence of no such f existing to deal with.

Theorem 4.2. *Let (M^n, g) be a complete, connected, non-compact Riemannian manifold. Then there exists a smooth function*

$$G : M \times M \setminus \operatorname{diag}(M) \to \mathbb{R}$$

satisfying the following:

1. *for $x \neq y$ in M we have $G(x,y) = G(y,x)$,*
2. *given $x \in M$ and $K \subseteq M$ compact with $x \in K$ there exists $C = C(K) > 0$ such that the three bullet points of (2) in theorem 4.1 hold,*
3. *for $\phi \in C_c^2(M)$, we have:*

$$\phi(x) = \int_M G(x,y)(\Delta\phi)(y)\mu_g(dy).$$

Corollary 4.3. *Let (M^n, g) be complete non-compact and connected. Then there exists a **positive** symmetric Green's function on (M,g) if and only if there exists a $p \in M$, $R > 0$ and component of $M \setminus B_R(p)$ which admits a non-constant bounded harmonic function f whose infimum on this component occurs at infinity.*

As an aside, the symmetric Greens function constructed from theorem 4.1 above has the property that for every $x \in M$ and $r > 0$ we have

$$\sup_{y \in M \setminus B_r(x)} G(x,y) = \sup_{y \in \partial B_r(x)} G(x,y).$$

However, in general, G is not bounded below.

5 Existence of Heat Kernel on a Manifold

As in the previous section, let (M^n, g) be complete and non-compact and choose an exhaustion by compact submanifolds with boundary:

$$\overline{\Omega_i} \subseteq \Omega_{i+1}, \quad M = \bigcup_{i=1}^{\infty} \Omega_i.$$

From spectral theory, we know there exists an orthonormal basis $\phi_{i,j}$ for $L^2(\Omega_i)$ (all real-valued) such that:

$$\begin{cases} -\Delta\phi_{i,j} & = \lambda_j(\Omega_i)\phi_{i,j} \\ \phi_{i,j}|_{\partial\Omega_i} & = 0 \end{cases}$$

with $0 < \lambda_1(\Omega_i) \leq \lambda_2(\Omega_i) \leq \cdots$. Furthermore, we know that $\lambda_j(\Omega_i) \to \infty$ as $j \to \infty$.

Note: If Ω_i is connected then $\lambda_1(\Omega_i) < \lambda_2(\Omega_i)$. Furthermore, for $t > 0$, $e^{t\Delta}$ is bounded and self-adjoint on $L^2(\Omega)$.

We now set

$$p_i(x, y, t) := \sum_{j=1}^{\infty} e^{-t\lambda_i(\Omega_i)} \phi_{i,j}(x)\phi_{i,j}(y)$$

and notice that the Weyl law

$$\lambda_j(\Omega_i) \sim c_n(j/\operatorname{Vol}(\Omega_i))^{2/n} \text{ as } j \to \infty$$

implies convergence of $p_i(x, y, t)$ in $C^k_{loc}(\Omega_i \times \Omega_i \times (0, \infty))$ for all $k \geq 0$.

Proposition 5.1. *For every $t > 0$ and $x, y \in \Omega_i$ we have $p_i(x, y, t) > 0$. Furthermore, $p_i \in C^\infty(\Omega_i \times \Omega_i \times (0, \infty))$.*

This is proven using the parabolic maximum principle. Namely, set

$$\Omega_i^T := \Omega_i \times (0, T) \text{ and } \partial_P\Omega_i^T \partial\Omega_i^T \setminus \{(x, T) : x \in \overline{\Omega_i}\}.$$

Then if $u \in C^0(\overline{\Omega_i}^T) \cap C^2(\Omega_i^T)$ solves $\partial_t u = \Delta u$ on Ω_i^T we have:

$$\sup_{\Omega_i^T} u = \sup_{\partial_P\Omega_i^T} u \text{ and } \inf_{\Omega_i^T} u = \inf_{\partial_P\Omega_i^T} u.$$

This also implies that for every $x \in \Omega_i$ and every $t > 0$ we have

$$\int_{\Omega_i} p_i(x, y, t)\mu_g(dy) \leq 1.$$

Corollary 5.2. *For all $x, y \in \Omega_i$, $t > 0$ we have*

$$p_i(x, y, t) \leq p_{i+1}(x, y, t).$$

Using this we can define

$$p(x, y, t) := \lim_{i\to\infty} p_i(x, y, t) \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

The key point is now that our above integral bound on $p_i(x, y, t)$ implies that for every $x \in M$ and $t > 0$ we have

$$\int_M p(x, y, t)\mu_g(dy) \leq 1$$

and so $p(x,y,t) < \infty$ almost everywhere! Furthermore, if $e^{t\Delta}$ is defined on $L^2(M)$ via spectral theory then $p(x,y,t)$ is an integral kernel for this operator!

Aside: Via a "monotonicity in t" theorem one can show that for every $x \in M$, $p(x,x,t)$ is monotone increasing in t for $t > 0$.

Theorem 5.3. *If (M^n, g) is complete with Ric_g bounded below then for every $x \in M$ and $t > 0$ we have*

$$\int_M p(x,y,t)\mu_g(dy) = 1.$$

*This conclusion is called **stochastic completeness**.*

6 Connections/Covariant Derivatives

Throughout, $U \subseteq \mathbb{R}^n$ is a connected open set. A **connection**, acting on $C^\infty(U, \mathbb{R}^k)$, is simply a collection of n smooth maps:

$$A_i : U \to M_k(\mathbb{R}), \;\; i = 1, ..., n$$

where $M_k(\mathbb{R})$ is the collection of all $k \times k$ matrices with entries in $\mathbb{R}$. We often write

$$A = A_1 dx^1 + \cdots + A_n dx^n$$

which, for the moment, is just a notation and we use A to denote this connection. For $\alpha, \beta \in \{1, ..., k\}$ we write the α, β-entry of the matrix $A_i(x)$ (α labeling the row, and β labeling the column) as $A^\alpha_{i,\beta}(x)$. Often we assume that:

$$A_i^T = -A_i \;\text{ for all } i.$$

Now, suppose $f : U \to \mathbb{R}^k$ is a smooth function. Let $f^1, ..., f^k : U \to \mathbb{R}$ denote the components of f. We define the **gradient** of f to be the matrix of functions with k rows and n columns given by

$$\nabla f := \begin{pmatrix} \nabla f^1 \\ \vdots \\ \nabla f^k \end{pmatrix}$$

where the gradients ∇f^i are interpreted as row vectors. For any smooth vector field

$$X : U \to \mathbb{R}^n$$

we can define a smooth function

$$\nabla_X f : U \to \mathbb{R}^k$$

by

$$\nabla_X f := \begin{pmatrix} \langle \nabla f^1, X \rangle \\ \vdots \\ \langle \nabla f^k, X \rangle \end{pmatrix}$$

where $\langle -, - \rangle$ denotes the dot product.

Definition 6.1. Given a connection A we define the **covariant derivative** associated to A to be the operator ∇^A given by

$$\begin{aligned} \nabla^A : C^\infty(U, \mathbb{R}^k) &\to C^\infty(U, M_{k,n}(\mathbb{R})) \\ \nabla^A f &:= \nabla f + \begin{pmatrix} A_1 f & \cdots & A_n f \end{pmatrix} \end{aligned}$$

where each $A_i f$ is a column vector in $\mathbb{R}^k$ so that the right hand side indeed has k rows and n columns. For any smooth vector field $X : U \to \mathbb{R}^n$ we denote

$$\nabla_X^A f := \nabla_X f + \sum_{i=1}^{n} (A_i f) X^i$$

where X^i is the i'th component of X.

Below we list several of the main properties of covariant derivatives.

Proposition 6.2. *Let A be a connection, $f, f_1, f_2 : U \to \mathbb{R}^k$ smooth, $a \in \mathbb{R}$, $h \in C^\infty(U)$ and $X, X_1, X_2 : U \to \mathbb{R}^n$ smooth vector fields. Then:*

1. $\nabla_X^A (f_1 + a f_2) = \nabla_X^A f_1 + a \nabla_X^A f_2$. *i.e.* ∇_X^A *is* $\mathbb{R}$*-linear.*
2. $\nabla_{X_1 + hX_2}^A f = \nabla_{X_1}^A f + h \nabla_{X_2}^A f$. *i.e.* $\nabla^A f$ *is* $C^\infty(U)$*-linear.*
3. *(product rule)* $\nabla_X^A (hf) = (\nabla_X h) f + h \nabla_X^A f$.
4. *Suppose* $A_1, ..., A_n$ *all satisfy* $A_i^T = -A_i$. *Then* $\nabla_X \langle f_1, f_2 \rangle = \langle \nabla_X^A f_1, f_2 \rangle + \langle f_1, \nabla_X^A f_2 \rangle$.

The main utility of connections comes from the following calculation.

Proposition 6.3. Gauge Equivariance
Suppose $f : U \to \mathbb{R}^k$ and $g : U \to \mathrm{GL}(\mathbb{R}^k)$ are smooth functions and A is a connection. For each $x \in U$ we can multiply the vector $f(x)$ into the matrix $g(x)$ to obtain a new vector and so we get a new function:

$$(gf)(x) := g(x) f(x) \in \mathbb{R}^k.$$

Writing $g^{-1} : U \to \mathrm{GL}(\mathbb{R}^k)$ for the function mapping x to the inverse matrix of $g(x)$, we then let $g(A)$ denote the new connection given by $g(A)_1, ..., g(A)_n : U \to M_k(\mathbb{R})$ where

$$g(A)_i := -(\partial_i g) g^{-1} + g A_i g^{-1} \;\; \text{for} \;\; i = 1, ..., n.$$

Then, for any smooth vector field $X : U \to \mathbb{R}^n$ we have

$$\nabla_X^{g(A)} (gf) = g \nabla_X f.$$

Proof. We simply compute:

$$\begin{aligned}
\nabla_X^{g(A)} (gf) &= \nabla_X (gf) + \sum_{i=1}^{n} X^i (-(\partial_i g) g^{-1} + g A_i g^{-1})(gf) \\
&= \sum_{i=1}^{n} \left(X^i \partial_i (gf) - X^i (\partial_i g) g^{-1} g f + X^i g A_i g^{-1} g f \right) \\
&= \sum_{i=1}^{n} \left(g X^i \partial_i f + g X^i A_i f \right) \\
&= g \left(\nabla_X f + \sum_{i=1}^{n} X^i A_i f \right) \\
&= g \nabla_X^A f.
\end{aligned}$$

$\square$

The point of the above identity is the following. The transformations such as g are to vector bundles what coordinate transformations are to smooth manifolds. So the above transformation law will tell us how a connection transforms under the vector-bundle-analogue to a change of coordinates.

For now, let's give an alternate characterization of a connection.

Proposition 6.4. *Suppose we had a map*

$$D : C^\infty(U, \mathbb{R}^k) \to C^\infty(U, M_{k,n}(\mathbb{R}))$$

where, for $f \in C^\infty(U, \mathbb{R}^k)$ and $X : U \to \mathbb{R}^n$ we write the product of the matrix Df with the vector X as $D_X f$. Furthermore, suppose that D satisfied the first three properties in proposition 6.2. Then there exists a unique connection A such that

$$D = \nabla^A.$$

So, as we'll see later, one can use the first three properties of proposition 6.2 to **define** a covariant derivative/connection on a vector bundle over a smooth manifold. Then, proposition 6.4 tells us that locally (i.e. in each local trivialization), our covariant derivative can be written as ∇^A for some unique matrix-valued function A. Finally, proposition 6.3 tells us how A changes when we perform a change of local trivialization.

7 Connections/Covariant Derivatives

Let's begin by defining covariant derivatives on $\mathbb{R}^n$. The case of a covariant derivative on a vector bundle over a manifold is locally the same as the $\mathbb{R}^n$ case. Throughout, as usual, $U \subseteq \mathbb{R}^n$ is a connected open set.

First I should define what an $\mathbb{R}^k$-valued *differential form* on U is. Let S_ℓ denote the group of all bijections $\sigma : \{1, ..., \ell\} \to \{1, ..., \ell\}$ and write

$$(-1)^\sigma := \begin{cases} 1 & \text{if } \sigma \in A_\ell = \text{ the alternating group,} \\ -1 & \text{if } \sigma \notin A_\ell. \end{cases}$$

For $\ell \in \mathbb{Z}_{\geq 1}$, we denote by:

$$\Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k$$

the set of all maps

$$T : (\mathbb{R}^n)^{\times \ell} = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^k$$

satisfying the following:

1. For any $i \in \{1, ..., \ell\}$,

$$T(\vec{v}_1, ..., \vec{v}_{i-1}, \vec{v}_i + a\vec{w}, \vec{v}_{i+1}, ..., \vec{v}_\ell) = T(\vec{v}_1, ..., \vec{v}_\ell) + aT(\vec{v}_1, ..., \vec{v}_{i-1}, \vec{w}, \vec{v}_{i+1}, ..., \vec{v}_\ell)$$

   for all $\vec{v}_1, ..., \vec{v}_\ell, \vec{w} \in \mathbb{R}^n$ and all $a \in \mathbb{R}$. That is to say, T takes in ℓ vectors in $\mathbb{R}^n$ as input, and is linear in each of these ℓ vectors separately.

2. For every $\vec{v}_1, ..., \vec{v}_\ell \in \mathbb{R}^n$ and every $\sigma \in S_\ell$ we have:

$$T(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(\ell)}) = (-1)^\sigma T(\vec{v}_1, ..., \vec{v}_\ell).$$

So, for example:

$$T(\vec{v}_1, \vec{v}_2, \vec{v}_3, ..., \vec{v}_\ell) = -T(\vec{v}_2, \vec{v}_1, \vec{v}_3, ..., \vec{v}_\ell).$$

There are several important examples of elements of $\Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k$ that we should discuss. As per usual, $\langle -, - \rangle$ will denote the dot product on $\mathbb{R}^n$. For $i = 1, ..., n$ and $\vec{u} \in \mathbb{R}^k$ fixed we have elements

$$f^i \otimes \vec{u} \in \Lambda^1(\mathbb{R}^n)^* \otimes \mathbb{R}^k$$

defined by

$$\begin{aligned} f^i &: \mathbb{R}^n \to \mathbb{R}^k \\ f^i(\vec{v}) &:= \langle \vec{e}_i, \vec{v} \rangle \vec{u} \end{aligned}$$

where $\vec{e}_i \in \mathbb{R}^n$ is the i'th standard basis vector.

As a bit of notation: if $k = 1$ then we write

$$\Lambda^\ell(\mathbb{R}^n)^* := \Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^1.$$

Next, if we have $T_a \in \Lambda^a(\mathbb{R}^n)^*$ and $T_b \in \Lambda^b(\mathbb{R}^n)^* \otimes \mathbb{R}^k$ we define a new element

$$T_a \wedge T_b \in \Lambda^{a+b}(\mathbb{R}^n)^* \otimes \mathbb{R}^k$$

by the formula

$$(T_a \wedge T_b)(\vec{v}_1, ..., \vec{v}_{a+b}) := \frac{1}{a!b!} \sum_{\sigma \in S_{a+b}} (-1)^\sigma T_a(\vec{v}_{\sigma(1)}, ..., \vec{v}_{\sigma(a)}) T_b(\vec{v}_{\sigma(a+1)}, ..., \vec{v}_{\sigma(a+b)}).$$

The main example to keep in mind here it that if $S, T \in \Lambda^1(\mathbb{R}^n)^*$ then

$$(S \wedge T)(\vec{v}, \vec{w}) = S(\vec{v})T(\vec{w}) - S(\vec{w})T(\vec{v}).$$

Lemma 7.1. *If $T_1, T_2 \in \Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k$ and $a \in \mathbb{R}$ then the map*

$$(T_1 + aT_2)(\vec{v}_1, ..., \vec{v}_\ell) := T_1(\vec{v}_1, ..., \vec{v}_\ell) + aT_2(\vec{v}_1, ..., \vec{v}_\ell)$$

is also in $\Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k$. Hence $\Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k$ is a real vector space. Furthermore:

$$\dim_{\mathbb{R}} \Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k = k\binom{n}{\ell}.$$

Since $\Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k$ is a finite dimensional real vector space, it makes sense to discuss smooth maps $f : U \to \Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k$. These are exactly what differential forms are.

Definition 7.2. A **differential ℓ-form** valued in $\mathbb{R}^k$ is a smooth function

$$\alpha : U \to \Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k.$$

Our notation for evaluating α at a point $x \in U$ to get an element of $\Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k$ is: $\alpha_x \in \Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k$. So, for $x \in U$ and $\vec{v}_1, ..., \vec{v}_\ell$ we have

$$\alpha_x(\vec{v}_1, ..., \vec{v}_\ell) \in \mathbb{R}^k.$$

We denote

$$\Omega^\ell(U; \mathbb{R}^k) := C^\infty(U, \Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k)$$

and when $k = 1$ we simply write $\Omega^\ell(U)$.

Fixing $\vec{u} \in \mathbb{R}^k$ there are several important $\mathbb{R}^k$-valued differential forms we should discuss. First of all, for $i = 1, ..., n$ we denote by

$$dx^i \otimes \vec{u}$$

the constant function

$$\begin{aligned} U &\to \Lambda^\ell(\mathbb{R}^n)^* \otimes \mathbb{R}^k \\ x &\mapsto f^i \otimes \vec{u}. \end{aligned}$$

Lemma 7.3. *Let $\alpha \in \Omega^\ell(U; \mathbb{R}^k)$ and write $\vec{\psi}_1, ..., \vec{\psi}_k$ for the standard basis for $\mathbb{R}^k$. Then there exists unique smooth functions $f^j_{i_1 \cdots i_\ell} : U \to \mathbb{R}$ for $j = 1, ..., k$ and $i_m \in\in \{1, ..., n\}$ satisfying*

$$f^j_{i_{\sigma(1)} \cdots i_{\sigma(\ell)}} = (-1)^\sigma f^j_{i_1 \cdots i_\ell}$$

such that

$$\alpha = \sum_{j=1}^{k} \sum_{i_1, ..., i_\ell = 1}^{n} f^j_{i_1 \cdots i_\ell} dx^{i_1} \wedge \cdots \wedge dx^{i_\ell} \otimes \vec{\psi}_j.$$

A covariant derivative is essentially just a generalization of the gradient operator ∇.

8 Seiberg-Witten: local theory

To avoid discussing annoying geometry stuff, I want to look at the Seiberg-Witten equations on the unit ball $B_1(0^n) \subseteq \mathbb{R}^n$.

Unfortunately, we need to discuss a bit about **spinors**. Recall that every Lie group G has a Lie algebra $\mathfrak{g}$. The representations of the Lie algebra $\mathfrak{g}$ can be computed using some (admittedly complicated) linear algebra. In particular, one can write computer code to classify all of them (for example, `http://www-math.univ-poitiers.fr/~maavl/LiE/form.html`). However, while taking derivatives shows that

every representation of G induces a representation of $\mathfrak{g}$,

the converse is not true! As we will see, spinors live in a representation of the Lie algebra

$$\mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) \ : \ X^T = -X\}$$

which **do not** come from representations of $\mathrm{SO}(n)$.

Theorem 8.1. *Let G be a connected Lie group. If $\pi_1(G) = 0$ then every representation of the Lie algebra $\mathfrak{g}$ is the derivative of a representation of $\mathfrak{g}$. More generally, if $\pi_1(G) \neq 0$ then let $\pi : \widetilde{G} \to G$ be the universal cover of G. Then $\widetilde{G}$ is a Lie group and the derivative of π at the identity element induces an isomorphism between the Lie algebra of $\widetilde{G}$ and $\mathfrak{g}$. Thus every representation of $\mathfrak{g}$ is the derivative of a representation of $\widetilde{G}$.*

If you only care about spinors in dimensions $n = 3, 4$ (which is the relevant case for Seiberg-Witten, at least for the moment), skip to 8.7!

Let's see what happens in the special case $G = \mathrm{SO}(n)$.

Throughout, I will assume $n \geq 3$. The Lie group $\mathrm{SO}(n)$ is connected, but not simply connected. Since $\mathrm{SO}(n)$ consists of $n \times n$ matrices, it acts on $\mathbb{R}^n$ and, since the matrices are orthogonal, it preserves the lengths of vectors. Thus we have an action:

$$\mathrm{SO}(n) \text{ acts on } S^{n-1} \subseteq \mathbb{R}^n.$$

Lemma 8.2. *The action of* $\mathrm{SO}(n)$ *on* S^{n-1} *is transitive. i.e. for any* $\vec{v}, \vec{w} \in S^{n-1}$*, there exists an* $A \in \mathrm{SO}(n)$ *such that* $A\vec{v} = \vec{w}$.

Lemma 8.3. *Consider the stabilizer of* $\vec{e}_1 \in S^{n-1}$*:*

$$\{A \in \mathrm{SO}(n) \ : \ A\vec{e}_1 = \vec{e}_1\}.$$

This is the set of all $A \in \mathrm{SO}(n)$ *such that*

$$A_{1i} = A_{i1} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

for all $i = 1, ..., n$. *This subgroup is closed in* $\mathrm{SO}(n)$ *and is isomorphic to* $\mathrm{SO}(n-1)$.

Lemma 8.4. *The quotient of* $\mathrm{SO}(n)$ *by the above stabilizer subgroup is diffeomorphic to* S^{n-1}.

So, if $\pi : \mathrm{SO}(n) \to S^{n-1}$ is the quotient map and if $i : \mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n)$ is the map

$$i(B) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}$$

then the *fibration*

$$\mathrm{SO}(n-1) \xrightarrow{i} \mathrm{SO}(n) \xrightarrow{\pi} S^{n-1}$$

gives rise to the long exact sequence in homotopy groups:

$$\cdots \to \pi_2(\mathrm{SO}(n)) \cdots \to \pi_2(S^{n-1}) \to \pi_1(\mathrm{SO}(n-1)) \to \pi_1(\mathrm{SO}(n)) \to \pi_1(S^{n-1}) \to 0.$$

To analyze this long exact sequence we'll need the following result.

Proposition 8.5. *For* $k < n$ *we have* $\pi_k(S^n) = 0$ *and for* $k = n$ *we have* $\pi_n(S^n) \cong \mathbb{Z}$.

Since we're assuming $n \geq 3$, we have $n - 1 \geq 2$ and so S^{n-1} is simply connected. When $n \geq 4$ we have $n - 1 \geq 3$ and so, in this case, we also have $\pi_2(S^{n-1}) = 0$. So, for $n \geq 4$ the following sequence is exact:

$$0 \to \pi_1(\mathrm{SO}(n-1)) \to \pi_1(\mathrm{SO}(n)) \to 0$$

thus

$$\pi_1(\mathrm{SO}(n-1)) \cong \pi_1(\mathrm{SO}(n)).$$

What happens when $n = 3$? In this case we the following tricky result.

Proposition 8.6. $\mathrm{SO}(3)$ *is diffeomorphic to* $\mathbb{R}P^3$.

Finally, recall that $\mathbb{R}P^3$ is the quotient of S^3 by the antipodal map. The quotient map $S^3 \to \mathbb{R}P^3$ is a covering space whose fibers each have exactly two elements. Thus $\pi_1(\mathbb{R}P^3)$ is a group with two elements. The only such group is $\mathbb{Z}/2$. Putting all of this together we arrive at the following.

Proposition 8.7. *For* $n \geq 3$, $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2$.

To avoid doing a bunch of algebra, let's stick to the most important cases for the Seiberg-Witten equations: $n = 3, 4$. Consider the Lie group

$$\mathrm{SU}(2) = \{A \in M_2(\mathbb{C}) \ : \ AA^* = A^*A = I_{2\times 2} \text{ and } \det(A) = 1\}$$

where $A^* := \overline{A}^T$ with $\overline{A}$ denoting complex conjugate of all entries.

Theorem 8.8. *The universal cover of* $\mathrm{SO}(4)$ *is* $\mathrm{SU}(2) \times \mathrm{SU}(2)$ *and the universal cover of* $\mathrm{SO}(3)$ *is* $\mathrm{SU}(2)$.

Proof. First, we notice that every element of $\mathrm{SU}(2)$ can be written uniquely as

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \text{ for } z, w, \in \mathbb{C} \text{ satisfying } |z|^2 + |w|^2 = 1.$$

Next, we define two actions of $\mathrm{SU}(2)$ on $\mathbb{R}^4$. First of all:

$$\begin{pmatrix} y_1 + iy_2 & y_3 + iy_4 \\ -y_3 + iy_4 & y_1 - iy_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} := \begin{pmatrix} y_1x_1 - y_2x_2 - y_3x_3 - y_4x_4 \\ y_1x_2 + y_2x_1 + y_3x_4 - y_4x_3 \\ y_1x_3 + y_3x_1 + y_4x_2 - y_2x_4 \\ y_1x_4 + y_4x_1 + y_2x_3 - y_3x_2 \end{pmatrix}$$

The other action we define is a right action:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 + iy_2 & y_3 + iy_4 \\ -y_3 + iy_4 & y_1 - iy_2 \end{pmatrix} := \begin{pmatrix} x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 \\ x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3 \\ x_1y_3 + x_3y_1 + x_4y_2 - x_2y_4 \\ x_1y_4 + x_4y_1 + x_2y_3 - x_3y_2 \end{pmatrix}.$$

Using these actions we define a homomorphism:

$$\begin{aligned} \kappa : \mathrm{SU}(2) \times \mathrm{SU}(2) &\to \mathrm{GL}(\mathbb{R}^4) \\ (A, B) &\mapsto [\vec{x} \mapsto A \cdot \vec{x} \cdot B^*]. \end{aligned}$$

There are several things we need to check, the first of which is that the image of κ is contained in $\mathrm{SO}(4)$ (in fact, the image will be equal to $\mathrm{SO}(4)$). To check this, it suffices to check it on elements of the form $(A, I_{2\times 2})$ and $(I_{2\times 2}, B)$ for elements $A, B \in \mathrm{SU}(2)$ of the form:

$$\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} \text{ for } |z| = 1, \ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

since these generate $\mathrm{SU}(2)$. This verification is left to the reader. So we now have that κ is a homomorphism:

$$\kappa : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4).$$

We will omit the proof that κ is surjective out of laziness.

Next, we need to compute the kernel of κ. The pair (A, B) is in the kernel if and only if $\kappa(A, B)\vec{x} = \vec{x}$ for all $\vec{x} \in \mathbb{R}^4$. So, let:

$$A = \begin{pmatrix} a_1 + ia_2 & a_3 + ia_4 \\ -a_3 + ia_4 & a_1 - ia_2 \end{pmatrix} \quad \text{and} \quad B^* = \begin{pmatrix} b_1 + ib_2 & b_3 + ib_4 \\ -b_3 + ib_4 & b_1 - ib_2 \end{pmatrix}$$

Note: the b_ℓ's are the entries of B^* not B!!!

We will check the identity $\kappa(A, B)\vec{x} = \vec{x}$ for the special case of $\vec{x} = \vec{e}_\ell$, the standard basis vectors. We compute:

$$\begin{aligned} \kappa(A, B)\vec{e}_1 &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 + ib_2 & b_3 + ib_4 \\ -b_3 + ib_4 & b_1 - ib_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 \\ a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3 \\ a_1b_3 + a_3b_1 + a_4b_2 - a_2b_4 \\ a_1b_4 + a_4b_1 + a_2b_3 - a_3b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Now, since $A, B \in \mathrm{SU}(2)$ we know that $b_1^2 + b_2^2 + b_3^2 + b_4^2 = 1$. So, let's multiply the first row by b_1, the second row by b_2, and add them together. The result is:

$$\begin{aligned}
b_1 &= a_1(b_1^2 + b_2^2) - a_2 b_2 b_1 - a_3 b_3 b_1 - a_4 b_4 b_1 + a_2 b_1 b_2 + a_3 b_2 b_4 - a_4 b_2 b_3 \\
&= a_1(b_1^2 + b_2^2) - b_3(a_3 b_1 + a_4 b_2) - b_4(a_4 b_1 - a_3 b_2) \\
&= a_1(b_1^2 + b_2^2) - b_3(-a_1 b_3 + a_2 b_4) - b_4(-a_1 b_4 - a_2 b_3) \\
&= a_1(b_1^2 + b_2^2) + a_1(b_3^2 + b_4^2) \\
&= a_1.
\end{aligned}$$

Similarly, we can multiply the first row by b_2 and subtract the second row times b_1 to obtain (after a similar computation) that $b_2 = -a_2$. The same type of argument also shows that $b_3 = -a_3$ and $b_4 = -a_4$. Thus:

$$(A, B) \text{ in the stabilizer of } \vec{e}_1 \iff B^* = \begin{pmatrix} a_1 - ia_2 & -a_3 - ia_4 \\ a_3 - ia_4 & a_1 + ia_2 \end{pmatrix} = A^*$$

thus $A = B$. In particular, if we let

$$\iota : \mathrm{SU}(2) \to \mathrm{SU}(2) \times \mathrm{SU}(2), \quad \iota(A) := (A, A)$$

then the composition $\kappa \circ \iota : \mathrm{SU}(2) \to \mathrm{SO}(4)$ has the stabilizer of $\vec{e}_1$ as its image. Since this stabilizer is isomorphic to $\mathrm{SO}(3)$ we see that $\kappa \circ \iota$ gives us a homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$.

Now, let's continue our computation with the $\kappa(A, B)\vec{e}_2 = \vec{e}_2$ equation. However, since we're computing the kernel, the $\vec{e}_1$-case shows us that we have $A = B$ and so we only need to analyze $\kappa(A, A)\vec{e}_2 = \vec{e}_2$. This equation can be computed out to be:

$$\begin{pmatrix} -a_2 a_1 - a_1 a_2 - a_4 a_3 + a_3 a_4 \\ -a_2^2 + a_1^2 + a_4^2 + a_3^2 \\ -a_2 a_3 + a_4 a_1 - a_3 a_2 - a_1 a_4 \\ -a_2 a_4 - a_3 a_1 + a_1 a_3 - a_4 a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The first of these equations becomes $a_1 a_2 = -a_1 a_2$ and therefore $a_1 a_2 = 0$. Similarly, the third and fourth of these equations yield $a_1 a_4 = 0 = a_1 a_3$. However, $A \in \mathrm{SU}(2)$ so we know that $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$ and we can combine this with the second equation to get $-2a_2^2 = 0$ hence $a_2 = 0$. Doing this for $\kappa(A, A)\vec{e}_3 = \vec{e}_3$ and $\kappa(A, A)\vec{e}_4 = \vec{e}_4$ only gives us two new equations:

$$\begin{aligned}
-a_3^2 + a_1^2 + a_2^2 + a_4^2 &= 1 \\
-a_4^2 + a_1^2 + a_2^2 + a_3^2 &= 1.
\end{aligned}$$

which, similar to the a_2 case, demonstrate that $a_3 = 0 = a_4$. All together, we're only left with one equation:

$$a_1^2 = 1, \quad \text{i.e. } a_1 = \pm 1.$$

So we see that

$$\ker(\kappa) = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} \cong \mathbb{Z}/2$$

as desired. Thus $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is indeed the double-cover of $\mathrm{SO}(4)$ and κ is our covering map. In the same way, $\kappa \circ \iota$ gives us a surjective homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$ with the exact same kernel as above ($\cong \mathbb{Z}/2$) so $\mathrm{SU}(2)$ is the universal cover of $\mathrm{SO}(3)$. □

Definition 8.9. The homomorphism

$$\begin{aligned}
\mathrm{SU}(2) \times \mathrm{SU}(2) &\to \mathrm{SU}(2) \subseteq \mathrm{GL}(\mathbb{C}^2) \\
(A, B) &\mapsto A
\end{aligned}$$

is called the **positive spinor representation** of $\mathfrak{so}(4)$ and the copy of $\mathbb{C}^2$ on the right hand side of the above is denoted by S_+. The **negative spinor representation** of $\mathfrak{so}(4)$ is:

$$\begin{aligned} \mathrm{SU}(2) \times \mathrm{SU}(2) &\to \mathrm{SU}(2) \subseteq \mathrm{GL}(\mathbb{C}^2) = \mathrm{GL}(S_-) \\ (A, B) &\mapsto B. \end{aligned}$$

Finally the representation of $\mathrm{SU}(2)$ given by the identity map

$$\mathrm{SU}(2) \to \mathrm{SU}(2) \subseteq \mathrm{GL}(\mathbb{C}^2)$$

is called the **spin representation** of $\mathfrak{so}(3)$ and the copy of $\mathbb{C}^2$ on the right hand side is denoted by S. **Note:** when people say **spinor** they usually mean either a smooth function

$$\psi : B_1(0^3) \to S$$

or a smooth function

$$\psi : B_1(0^4) \to S_+ \text{ or } \psi : B_1(0^4) \to S_-.$$

9 Sheaves

Here I would like to give an example of a sheaf that is not (explicitly) a sheaf of functions. This is perhaps **the** most important example for algebraic geometers and number theorists.

Definition 9.1. Let R be a commutative unital ring. Recall the following:

1. An **ideal** in R is a subset $I \subseteq R$ which contains $0 \in R$, is closed under addition, closed under subtraction, and has the property that for every $s \in I$ and every $r \in R$ we have $rs \in I$.
2. A **prime ideal** in R is a *proper* ideal $P \subsetneq R$ with the following additional property: if $a, b \in R$ and $ab \in P$ then at least one of a or b must be in P.
3. We denote
$$\mathrm{Spec}(R) := \{P \subseteq R \;:\; P \text{ is a prime ideal}\}.$$

In any ring, I will denote the zero ideal $\{0\}$ by either $\langle 0 \rangle$ or simply 0 (or sometimes (0)).

Example 9.2. Consider $R = \mathbb{Z}$. For any ideal I in R there exists a unique $n \in \mathbb{Z}_{\geq 0}$ such that

$$I = \mathbb{Z}n = \{nz \;:\; z \in \mathbb{Z}\}.$$

Why is this? Well, first of all notice that if $I = R$ then we get $n = 1$ since $\mathbb{Z}1 = \mathbb{Z}$ and if $n \neq 1$ then $n - 1 \notin I$, contradicting $I = R$. Next, we claim that:

$$n = \min\{|z| \;:\; z \in I \setminus \{0\}\}.$$

This minimum exists in $\mathbb{Z}_{\geq 1}$ since $\mathbb{Z}_{\geq 1}$ is bounded below. Furthermore, if $z \in I$ is the element such that $n = |z|$ then we have two cases: if $z \geq 0$ then $n = z$ and so $n \in I$. Otherwise, $z < 0$ and here $n = -z = (-1)z$ but this is in I since $z \in I$ and $-1 \in \mathbb{Z}$. Thus $n \in I$ and so

$$\mathbb{Z}n \subseteq I.$$

Now, let $y \in I$ be arbitrary. By the definition of n we have $n \leq |y|$ and so by long division there exists a $k \in \mathbb{Z}_{\geq 1}$ and $r \in \{0, ..., n-1\}$ such that

$$|y| = kn + r.$$

In either of the cases $y \geq 0$ or $y < 0$ we have $|y| \in I$. Since $n \in I$ and $k \in \mathbb{Z}$ we also have $kn \in I$ since I is an ideal. Therefore

$$r = |y| - kn \in I.$$

But $r \in \{0, ..., n-1\}$ thus $r \in I$ and $|r| < n$. Therefore, by the definition of n, we must have $r = 0$ and so $|y| = kn \in \mathbb{Z}n$. Therefore $y = \pm|y| \in \mathbb{Z}n$ as well. Since $y \in I$ was arbitrary we have arrived at

$$I \subseteq \mathbb{Z}n \text{ hence } I = \mathbb{Z}n.$$

Finally, to get uniqueness of n notice that if

$$\mathbb{Z}n = \mathbb{Z}m, \text{ for some } m \in \mathbb{Z}_{\geq 0}$$

then $m \in I \setminus \{0\}$ hence $n \leq |m| = m$. However, there must exist some $k \in \mathbb{Z} \setminus \{0\}$ such that $n = km$ since $n \in \mathbb{Z}m$. Thus

$$km = km = n \text{ therefore } m \leq n \text{ and so } n = m$$

as desired.

Okay, so we've now proven our classification of all ideals in $\mathbb{Z}$. What about prime ideals?

The ideal $\mathbb{Z}n$ is prime if and only if n is zero or a prime number (you can prove this with the fundamental theorem of arithmetic). Therefore:

$$\operatorname{Spec}(\mathbb{Z}) = \{\mathbb{Z}p \ : \ p \in \mathbb{Z}_{\geq 0} \text{ is either 0 or prime}\}.$$

For the next example we will need some lemmas.

Definition 9.3. Since every ring R is an abelian group under addition and every ideal I in R is a additive subgroup, we can form the quotient group R/I. Given $x \in R$, we write its I-coset as

$$x + I \in R/I.$$

Lemma 9.4. *For R commutative unital and I an ideal of R, the quotient R/I is also a ring with multiplicative identity given by $1 + I$ and multiplication defined by:*

$$(x + I)(y + I) := xy + I.$$

We will denote the additive identity $0 + I$ simply by 0 out of laziness.

Lemma 9.5. *Let R be a commutative unital ring and $I \trianglelefteq R$ an ideal. Then I is prime if and only if the quotient R/I is an integral domain (i.e. for every $x + I, y + I \in R/I$, if $(x + I)(y + I) = 0$ then either $x + I = 0$ or $y + I = 0$).*

Example 9.6. Let's look at the ring $R = \mathbb{Z}/n$. First, we claim: for any ideal $I \trianglelefteq \mathbb{Z}/n$, there exists a divisor m of n such that

$$I = (\mathbb{Z}/n)m = \{[zm] \in \mathbb{Z}/n \ : \ z \in \mathbb{Z}\}.$$

Indeed, I is in particular an additive subgroup of $\mathbb{Z}/n$ and we know that every additive subgroup is of this form. Next, the second isomorphism theorem for rings (I think) tells us that

$$R/I \cong \mathbb{Z}/m.$$

But now, $\mathbb{Z}/m$ is an integral domain if and only if m is prime. Indeed, if m is prime then this is a field so we're done. If m is not prime then $m = ab$ for some $a, b \in \{2, ..., m-1\}$. But now both $[a] \neq 0$ and $[b] \neq 0$ in $\mathbb{Z}/m$ meanwhile $[a][b] = [ab] = [m] = 0$ in $\mathbb{Z}/m$ therefore $\mathbb{Z}/m$ is not an integral domain. In conclusion:

$$\operatorname{Spec}(\mathbb{Z}/n) = \{(\mathbb{Z}/n)p \ : \ p \in \{2, ..., n-1\} \text{ is prime and } p \text{ divides } n\} \cup \{\langle 0 \rangle\}.$$

Example 9.7. Let's consider the ring

$$R = \mathbb{Z}[i] = \{a + bi \ : \ a, b \in \mathbb{Z}\}$$

of Gaussian integers. Here I will start to simply cite some results from Dummy and Cripple. First of all, $\mathbb{Z}[i]$ is a Euclidean domain with *norm function* given by

$$N(a + bi) := a^2 + b^2.$$

As such, $\mathbb{Z}[i]$ is a PID and a UFD, so all ideals I are of the form

$$I = \mathbb{Z}[i](a + ib) = \{(c + id)(a + ib) \in \mathbb{Z}[i] \ : \ c, d \in \mathbb{Z}\}$$

for some $a + ib \in \mathbb{Z}[i]$ and such an ideal is prime if and only if the element $a + ib$ is irreducible. Using the norm function there is a well-known classification of such irreducible elements. Namely, $a + ib$ is irreducible if and only if one of the following holds:

1. $a = 0$, $|b| = 4n + 3$ for some $n \in \mathbb{Z}_{\geq 0}$, and $|b|$ is prime in $\mathbb{Z}$,
2. $b = 0$, $|a| = 4n + 3$ for some $n \in \mathbb{Z}_{\geq 0}$, and $|a|$ is prime in $\mathbb{Z}$,
3. $a \neq 0$, $b \neq 0$, and $N(a + ib) = a^2 + b^2$ is prime in $\mathbb{Z}$.

For the next example, we recall the following theorem.

Theorem 9.8. *For any field k, the polynomial ring $k[x]$ is a principal ideal domain (PID). i.e. for any ideal I in $k[x]$, there exists $p(x) \in k[x]$ such that $I = (p(x))$ (here $(p(x))$ is the notation we use for the ideal generated by $p(x)$).*

For the analysis below, we will also need the following notion.

Definition 9.9. Let R be commutative unital and $\mathfrak{m} \subsetneq R$ a proper ideal. Then $\mathfrak{m}$ is called a **maximal ideal** if and only if whenever $I \subsetneq R$ is a proper ideal with $\mathfrak{m} \subseteq I$ we must have $\mathfrak{m} = I$.

Example 9.10. Let's consider the ring

$$R = \mathbb{C}[x] = \{ \text{ all polynomials in } x \text{ with coefficients in } \mathbb{C}\}.$$

By the above theorem, we know that $\mathbb{C}[x]$ is a PID. So, let $I = (p(x))$ be an ideal. By the fundamental theorem of algebra we can factor $p(x)$ as:

$$p(x) = (x - a_1) \cdots (x - a_d)$$

where d is the degree of $p(x)$ and $a_1, ..., a_d \in \mathbb{C}$ are its roots (listed with repetition). Now, for any two non-zero polynomials $f(x)$ and $g(x)$ we have

$$\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$$

and

$$\deg(f(x) + g(x)) = \max\{\deg(f(x)), \deg(g(x))\}.$$

Therefore every polynomial in $I = (p(x))$ has degree at least d. Therefore, the polynomials

$$f(x) = x - a_1 \ \text{ and } \ g(x) = (x - a_2) \cdots (x - a_d)$$

are both **not** in I and yet $f(x)g(x) = p(x) \in I$. Thus, unless $d = 1$ and so $g(x)$ doesn't exist, the ideal I is not prime. Therefore the only remaining candidates for prime ideals in $\mathbb{C}[x]$ are the ideals:

$$I = (x - a) \subseteq \mathbb{C}[x] \ \text{ for } \ a \in \mathbb{C}.$$

Notice that these ideals are indeed all prime (in fact, they are maximal) since the ideal $(x-a)$ is the kernel of the surjective homomorphism

$$\begin{aligned} \mathbb{C}[x] &\to \mathbb{C} \\ h(x) &\mapsto h(a) \end{aligned}$$

thus by the first isomorphism theorem for rings we have

$$\mathbb{C}[x]/(x-a) \cong \mathbb{C}$$

and $\mathbb{C}$ is an integral domain, hence $(x-a)$ is a prime ideal. Therefore:

$$\operatorname{Spec}(\mathbb{C}[x]) = \{(x-a) \ : \ a \in \mathbb{C}\} \cong \mathbb{C}.$$

At the moment this is just a bijection with $\mathbb{C}$, but in fact this is a homeomorphism if you give both $\operatorname{Spec}(\mathbb{C}[x])$ and $\mathbb{C}$ their *Zariski topologies*. Also, notice that in this example every prime ideal is maximal. That was not the case for $\mathbb{Z}$ and it won't be the case in the next example either.

Theorem 9.11. *If R is a UFD then the polynomial ring $R[x]$ is a UFD.*

Lemma 9.12. *Let R be commutative unital and $\mathfrak{m}$ an ideal. Then $\mathfrak{m}$ is maximal if and only if $R/\mathfrak{m}$ is a field.*

Example 9.13. Let's consider the ring

$$R = \mathbb{C}[x, y]$$

of all polynomials in two variables: x, y, with coefficients in $\mathbb{C}$. By the above theorem this is a UFD, but it is our first example which is not a PID! So not every ideal in $\mathbb{C}[x, y]$ is generated by a single element.

Let's first figure out what the maximal ideals look like. For any $a, b \in \mathbb{C}$ we have a surjective homomorphism

$$\begin{aligned} \mathbb{C}[x, y] &\to \mathbb{C} \\ f(x, y) &\mapsto f(a, b) \end{aligned}$$

whose kernel is the ideal with two generators:

$$(x-a, y-b) \trianglelefteq \mathbb{C}[x, y].$$

Therefore, by the first isomorphism theorem for rings and the above lemma we have that $(x-a, y-b)$ is maximal.

In fact, we claim that for any maximal ideal $\mathfrak{m}$ of $\mathbb{C}[x, y]$ there exists unique $a, b \in \mathbb{C}$ such that

$$\mathfrak{m} = (x-a, y-b).$$

The set $\operatorname{Spec}(R)$ for a general commutative unital ring R is not merely a set: there is a natural topology on it called the *Zariski topology*.

Definition 9.14. We say a subset $X \subseteq \operatorname{Spec}(R)$ is closed in the **Zariski topology** if and only if there exists an ideal $I \subseteq R$ such that

$$X = V(I) := \{P \in \operatorname{Spec}(R) \ : \ I \subseteq P\}.$$

A subset $U \subseteq \operatorname{Spec}(R)$ is called open in the Zariski topology if and only if its complement U^c is closed.

Proposition 9.15. *The collection of open subsets of* Spec(R)*, as defined above, actually define a topology on* Spec(R) *(i.e. they satisfy the axioms of a topology). This is called the Zariski topology.*

Proposition 9.16. *For each $f \in R$, the subset:*

$$D(f) := \{P \in \mathrm{Spec}(R) \ : \ f \notin P\}$$

is open. Furthermore, the collection of open sets $D(f)$ for every $f \in R$ is a basis for the Zariski topology on Spec(R).

So, given a commutative unital ring R we get a topological space Spec(R). We want to turn this into something called a "locally ringed space". To do this, we need to give it a structure sheaf so let's start laying the groundwork of how that is defined.

Definition 9.17. Let R be a commutative unital ring and $S \subseteq R$ a subset which is closed under multiplication and has $1 \in S$. We define the **localization** of R along S to be the ring denoted by $S^{-1}R$ and defined as follows. First, consider the set

$$X := \{(a, b) \ : \ a \in R, \ b \in S\}.$$

We define an equivalence relation $\sim$ on X by:

$$(a_1, b_1) \sim (a_2, b_2) \text{ if and only if there exists } s \in S \text{ such that } s(b_2 a_1 - b_1 a_2) = 0.$$

We then set:

$$S^{-1}R := X/\sim$$

and we denote the $\sim$-equivalence class of the pair (a, b) by $ab^{-1} \in S^{-1}R$. We can now define the various ring operations on $S^{-1}R$ by:

1. $a_1 b_1^{-1} + a_2 b_2^{-1} := (b_2 a_1 + b_1 a_2)(b_1 b_2)^{-1}$,
2. $(a_1 b_1^{-1}) \cdot (a_2 b_2^{-1}) := (a_1 a_2)(b_1 b_2)^{-1}$.

RECALL: When I say "unit" in ring theory, I mean any element which has a multiplicative inverse. I do not mean the identity element!

Proposition 9.18. *Let S, R and $S^{-1}R$ be as in the above definition. Then:*

1. *The ring operations $+, \cdot$ are well-defined. Furthermore, $01^{-1} \in S^{-1}R$ is an additive identity and $11^{-1} \in S^{-1}R$ is a multiplicative identity.*
2. *There exists additive inverses for all elements and they are given by: $-(ab^{-1}) = (-a)b^{-1}$.*
3. *There exists multiplicative inverses for all elements of the form $s1^{-1}$ with $s \in S$, and they are given by $(s1^{-1})^{-1} = 1s^{-1}$.*
4. *The map $\varphi : R \to S^{-1}R$ given by $\varphi(r) := r1^{-1}$ is a ring homomorphism.*
5. ***(Universal Property of Localization)** If $f : R \to T$ is any ring homomorphism such that $f(s)$ is a unit in T for every $s \in S$ then there exists a unique ring homomorphism $\widetilde{f} : S^{-1}R \to T$ such that $\widetilde{f} \circ \varphi = f$.*

Definition 9.19. In the special case where $S = R \setminus P$ for some prime ideal P we denote $R_P := S^{-1}R$ and call this the **localization of** R **at** P. Similarly, suppose we had an element $s \in R$ such that

$$S = \{1, s, s^2, s^3, ...\}.$$

Then we denote $R_s := S^{-1}R$ and call this the **localization of** R **away from** s.

Important Note: Consider the special case where $R = \mathbb{Z}$. We can then localize along the prime ideal $\mathbb{Z}2$ to get:

$$\mathbb{Z}_{\mathbb{Z}2} \cong \{a/b \in \mathbb{Q} \,:\, a, b \in \mathbb{Z},\ \gcd(a,b) = 1,\ b \neq 0, \text{ 2 does not divide } b\}.$$

On the other hand, we have:

$$\mathbb{Z}_2 \cong \{a/2^k \in \mathbb{Q} \,:\, a \in \mathbb{Z},\ k \in \mathbb{Z}\}.$$

Notice that these two rings are very much **not** the same! (Also: notice the terrible notation $\mathbb{Z}_2$!!! This is unfortunately the same notation as the ring of 2-adic integers despite being completely different from that ring!)

Proposition 9.20. *Let R be a commutative unital ring, $P \in \operatorname{Spec}(R)$ a prime ideal and $f \in R$ such that $f \notin P$. Then the map*

$$\begin{aligned} \varphi_{f,P} : R_f &\to R_P \\ rf^{-k} &\mapsto rf^{-k} \end{aligned}$$

is a ring homomorphism.

Definition 9.21. Let $X := \operatorname{Spec}(R)$. We will now define a sheaf $\mathcal{O}_X$ on the topological space X called the **structure sheaf** of X. For $U \subseteq \operatorname{Spec}(R)$ open we let:

$$\mathcal{O}_X(U) := \text{ the set of all functions } s : U \to \bigsqcup_{P \in U} R_P \text{ satisfying the following:}$$

1. $s(P) \in R_P$ for all $P \in U$,
2. for every $P \in U$ there exists a $f \in R$ and an element $rf^{-k} \in R_f$ such that

$$P \in D(f) \subseteq U$$

and

$$s(Q) = \varphi_{f,Q}(rf^{-k}) \text{ for every } Q \in D(f).$$

Theorem 9.22. *For $X = \operatorname{Spec}(R)$, $\mathcal{O}_X$ as defined above is a sheaf.*